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Estimates for the Modulus of Smoothness

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1. INTRODUCTION

For some time now, the modulus of smoothness has been used by approximation theorists as a neat measure of the structural properties of a function. For example, most theoretical estimates for the order of approximation of functions by say, polynomials or splines, are now given in terms of such a modulus.

Such theoretically elegant results are, however, not always easy to use in practice. Very often the potential user will have a particular function, and will require a *quantitative* estimate of its order of approximation. Such an estimate depends in turn on an accurate calculation of the order of the modulus of smoothness of the given function. This is often a difficult task, even in the relatively simple case of a first order modulus of a univariate function under the uniform norm. For high order moduli of multivariate functions with respect to say, L_q-norms, such a task becomes virtually impossible. The difficulties are compounded by the fact that, although we can often get some sort of *estimate* for the order of the modulus, it is usually very difficult to make sure that our estimate is sharp.

In this paper we shall introduce certain classes of multivariate functions which are characterised by the singularities of their elements. For functions from these classes we shall obtain sharp estimates of the modulus of smoothness of any order with respect to both the L_q- and uniform norms. In doing so we make the theoretical approximation theory for these classes of functions more accessible to the practical user.

The classes of functions which we will be able to handle will include those which contain a finite number of algebraic or logarithmic singularities. For example, let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a suitably smooth boundary. Suppose Ω contains the zero vector, let $\alpha > 0$, $1 \leqslant q \leqslant \infty$, and define $f\colon \Omega \to \mathbb{C}$ by

$$f(t) = |t|^{\alpha - n/q},$$

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where $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^n$. Then in Theorem 5 of Section 3, we shall prove that

$$\omega_k(f, \tau)_q = \begin{cases} O(\tau^\alpha), & k \in \mathbb{N}, k > \alpha, \\ O(\tau^k), & k \in \mathbb{N}, k < \alpha, \end{cases} \tag{1.1}$$

where $\omega_k(f, \tau)_q$ denotes the kth order modulus of smoothness of f in $L_q(\Omega)$, with parameter $\tau > 0$.

Theorem 5 also contains much more general results than those given in this example. In fact estimate (1.1) can still be obtained when f contains a number of algebraic or logarithmic singularities, provided the *dominant* singularity in f is $|t|^{\alpha - n/q}$.

The relevance of results like (1.1) becomes clearer when we look at some recent developments in approximation theory. If $Q \subseteq \mathbb{R}^n$ is a cube with side length τ then Brudnyi [5] has shown that for each $f \in L_q(Q)$, $1 \leqslant q \leqslant \infty$, there exists a multivariate polynomial p of total degree $\leqslant k-1$ such that

$$\|f - p\|_q \leqslant C\omega_k(f, \tau)_q, \tag{1.2}$$

where the norm and the modulus are evaluated in $L_q(Q)$, and C depends only on k and n. The estimate (1.2) leads to analogous results for piecewise polynomial approximation. For example (see Brudnyi [6]), if $\Omega = [0, 1]^n$, and Π denotes the partition of Ω into equal cubes of side length τ, then for any $f \in L_q(\Omega)$, there is a function p on Ω which is polynomial of degree $\leqslant k-1$ on each of the cubes in Π, and is such that

$$\|f - p\|_q \leqslant C\omega_k(f, \tau)_q. \tag{1.3}$$

In (1.3) C again depends only on k and n, and the norm and modulus are evaluated in $L_q(\Omega)$.

In fact, the literature contains a long tradition of approximation theoretic results like (1.2) and (1.3). A good review of a variety of results for the univariate case is given by De Vore [8], while spline approximation of multivariate functions has been developed by de Boor and Fix [2], Dahmen, De Vore, and Scherer [7], Munteanu and Schumaker [15], as well as in the aforementioned papers by Brudnyi [5, 6]. The developments of this paper show how estimates like those in the literature may be quantified for a large class of given functions f.

The main results of the paper are proved in Section 3. In Section 2 we introduce the modulus of smoothness and briefly review its properties. We also introduce in Section 2 a certain class of Banach function spaces—the Nikol'skii spaces—which are characterised by the behaviour of the moduli of smoothness of their members. The properties of these spaces are used in the development of Section 3. In Section 4 we calculate numerically the moduli of smoothness of some typical functions. The results show that, up to machine accuracy, the estimates of Section 3 are, in general, sharp.

In the literature some attention has been given to the practical calculation of the modulus of smoothness. Brenner, Thomée, and Wahlbin [4] give two examples of typical elements of univariate Besov spaces. Since the Nikol'skii spaces discussed in Section 2 are particular cases of Besov spaces, the examples in [4] can be viewed as statements about the moduli of smoothness of certain functions. In Trebels [19] (and in the references given there), results are obtained on the modulus of smoothness of functions whose Fourier transforms are known.

Estimates of the type proved in this paper have been used in [10, 11] to obtain convergence results for the solution of one- and two-dimensional integral equations using spline bases.

2. Modulus of Smoothness and Properties

The best surveys of properties of the modulus of smoothness are to be found in [12, 13, 16, 18]. We shall briefly summarise the important properties using the notation of [13].

Let $\mathbb{N}$ denote the positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout the paper Ω will denote a bounded domain in $\mathbb{R}^n$. Thus when $n = 1$, Ω is simply an open interval. We shall set

$$D = \sup_{x, y \in \Omega} |x - y|$$

(i.e., D is the diameter of Ω). Some of the results of this paper will require additional conditions on Ω; these will be stated when they are required. Let θ denote the zero vector in $\mathbb{R}^n$.

We denote by $C(\Omega)$ the set of all functions which are defined and continuous on Ω. We let $L_q(\Omega)$ $(1 \leqslant q \leqslant \infty)$ denote the usual Lebesgue space on Ω, and let $C(\bar{\Omega})$ denote the space of functions which are bounded and uniformly continuous on Ω. We note that $L_q(\Omega)$ and $C(\bar{\Omega})$ are Banach spaces and we denote their norms by $\|\cdot\|_{q,\Omega}$ and $\|\cdot\|_{\infty,\Omega}$, respectively, or just $\|\cdot\|_q$ and $\|\cdot\|_\infty$, if Ω is understood. For the multiindex $\beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^n$, of degree $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$, we use D^β to denote the differential operator $(\partial/\partial x_1)^{\beta_1} \cdots (\partial/\partial x_n)^{\beta_n}$, all derivatives being in the distributional sense [1]. Let $r \in \mathbb{N}$, $1 \leqslant q \leqslant \infty$. We define $W_q^r(\Omega)$ to be the (Sobolev) space of all functions f such that $D^\beta f \in L_q(\Omega)$, for all $0 \leqslant |\beta| \leqslant r$, and $C^r(\bar{\Omega})$ to be the space of all functions f with the property that $D^\beta f \in C(\bar{\Omega})$, for all $0 \leqslant |\beta| \leqslant r$. Then $W_q^r(\Omega)$ is a Banach space under the norm

$$\|f\|_{r,q} = \left\{ \sum_{0 \leqslant |\beta| \leqslant r} \|D^\beta f\|_q^q \right\}^{1/q},$$

and $C^r(\bar{\Omega})$ is a Banach space under the norm

$$\|f\|_{r,\infty} = \max_{0 \leqslant |\beta| \leqslant r} \|D^\beta f\|_\infty,$$

(see [1, pp. 9, 44–45]).

For $r \in \mathbb{N}_0$, $1 \leqslant q \leqslant \infty$, we then define (see [13]) $H^r_q(\Omega)$ as follows.

$$H^0_q(\Omega) = L_q(\Omega), \qquad 1 \leqslant q < \infty, \qquad H^0_\infty(\Omega) = C(\bar{\Omega}),$$
$$H^r_q(\Omega) = W^r_q(\Omega), \qquad 1 \leqslant q < \infty, \qquad H^r_\infty(\Omega) = C^r(\bar{\Omega}).$$

For any $h \in \mathbb{R}^n$, we define

$$\Omega_h = \{t \in \Omega : t + \delta h \in \Omega \text{ for } 0 \leqslant \delta \leqslant 1\}.$$

Then, for functions defined on Ω and $k \in \mathbb{N}_0$, we can define, for $t \in \Omega_{kh}$, the kth order forward difference of f

$$\Delta^k_h f(t) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} f(t + rh).$$

(Note that when $k = 1$ we have just $\Delta_h f(t) = f(t+h) - f(t)$.) For $1 \leqslant q \leqslant \infty$, $k \in \mathbb{N}_0$, the kth order modulus of smoothness is then a function

$$\omega_k : H^0_q(\Omega) \times (0, \infty) \to [0, \infty)$$

given by

$$\omega_k(f, \tau)_q = \sup_{0 < |h| \leqslant \tau} \|\Delta^k_h f\|_{q, \Omega_{kh}}.$$

Among the properties of the modulus of smoothness, we then have the following.

(i) For fixed $f \in H^0_q(\Omega)$, $k \in \mathbb{N}$, $1 \leqslant q \leqslant \infty$, $\omega_k(f, \tau)_q$ is a nondecreasing function of τ satisfying

$$\omega_k(f, \tau)_q \to 0 \qquad \text{as} \quad \tau \to 0,$$

for any $k \in \mathbb{N}$.

(ii) For fixed $\tau > 0$, $k \in \mathbb{N}_0$, we have

$$\omega_k(f_1 + f_2, \tau)_q \leqslant \omega_k(f_1, \tau)_q + \omega_k(f_2, \tau)_q,$$

for all $f_1, f_2 \in H^0_q(\Omega)$.

(iii) For fixed $f \in H^0_q(\Omega)$, $\tau > 0$, $k \in \mathbb{N}_0$, we have

$$\omega_k(f, \tau)_q \leqslant 2^j \omega_{k-j}(f, \tau)_q, \qquad 0 \leqslant j \leqslant k. \tag{2.1}$$

(iv) For $k \in \mathbb{N}$, $f \in H_q^j(\Omega)$ with $1 \leqslant j \leqslant k$, and $\tau > 0$, we have

$$\omega_k(f, \tau)_q \leqslant C\tau^j \sup_{|\beta| = j} \omega_{k-j}(D^\beta f, \tau)_q \tag{2.2}$$

with C independent of τ.

Throughout this paper, we shall have occasion to use statements of the form

$$f(\tau) = O(\phi(\tau)),$$

where $f(\tau)$ and $\phi(\tau)$ are nonnegative functions of $\tau > 0$. This will always mean that there exists a constant C (independent of τ), such that

$$f(\tau) \leqslant C\phi(\tau), \qquad \tau > 0.$$

Now, for $\alpha > 0$ and $1 \leqslant q \leqslant \infty$, we introduce (see [16, p. 159]) the Nikol'skii space $N_q^\alpha(\Omega)$. Let $r, \rho \in \mathbb{N}_0$ be such that $r > \alpha - \rho > 0$. We call (r, ρ) an *admissible pair for* α. Then, by definition, $f \in N_q^\alpha(\Omega)$ if $f \in L_q(\Omega)$, and there exists a constant M such that

$$\|\Delta_h^r D^\beta f\|_{q, \Omega_{rh}} \leqslant M |h|^{\alpha - \rho}, \tag{2.3}$$

for all multi-indices β with $|\beta| = \rho$, and all $h \in \mathbb{R}^n$. Then $N_q^\alpha(\Omega)$ is a Banach space under the norm

$$\|f\|_{\alpha, q} = \|f\|_q + |f|_{\alpha, q},$$

with the seminorm $|f|_{\alpha, q}$ given by

$$|f|_{\alpha, q} = \inf M,$$

where M is the constant appearing in (2.3), and the infimum is taken over all values of M for which (2.3) is satisfied.

It is not hard to see that the elements of $N_q^\alpha(\Omega)$ are characterised by the behaviour of their moduli of smoothness. In fact $f \in N_q^\alpha(\Omega)$ if and only if

$$\omega_r(D^\beta f, \tau)_q = O(\tau^{\alpha - \rho}) \tag{2.4}$$

for all $|\beta| = \rho$, where (r, ρ) is any admissible pair for α.

To clarify the definition of $N_q^\alpha(\Omega)$ further, we look at the simplest example of an admissible pair for α. If we let $[\alpha] \in \mathbb{N}_0$, and $0 < \alpha_0 \leqslant 1$ be such that

$$\alpha = [\alpha] + \alpha_0, \tag{2.5}$$

then if $0<\alpha_0<1$ $(1, [\alpha])$ is an admissible pair, whereas if $\alpha_0=1$, $(2, [\alpha])$ is an admissible pair. This leads to the observation that $f\in N_q^\alpha(\Omega)$ if and only if

$$\omega_1(D^\beta f, \tau)_q = O(\tau^{\alpha_0}), \qquad |\beta|=[\alpha], 0<\alpha_0<1,$$

$$\omega_2(D^\beta f, \tau)_q = O(\tau), \qquad |\beta|=[\alpha], \alpha_0=1.$$

The definition of $\|\cdot\|_{\alpha,q}$ given above depends on the choice of admissible pair. However it is shown by Nikol'skii in [16] that norms springing from distinct admissible pairs are pairwise equivalent provided the condition (i) of the following theorem in satisfied. We shall always assume that condition (i) is satisfied when we use $N_q^\alpha(\Omega)$.

THEOREM (Nikol'skii's imbedding theorem). (i) *Suppose the boundary Γ of Ω has the following property. For any $x_0\in\Gamma$ there exists a rectangular coordinate system $(\xi_1,..., \xi_n)$ with origin at x_0 and a cube*

$$\Delta := \{|\xi_j|<\eta_j : j=1,..., n\},$$

such that $\Delta\cap\Gamma$ may be described by an equation

$$\xi_n=\psi(\lambda),$$

for

$$\lambda\in\Delta' := \{(\xi_1,..., \xi_{n-1}) : |\xi_j|<\eta_j, j=1,..., n-1\},$$

where ψ satisfies the Lipschitz condition

$$|\psi(\lambda_1)-\psi(\lambda_2)|\leqslant C\,|\lambda_1-\lambda_2|, \qquad \lambda_1, \lambda_2\in\Delta',$$

and C is independent of λ_1, λ_2.

(ii) *Let $1\leqslant p\leqslant q\leqslant\infty$ and $\beta=\alpha-n(1/p-1/q)>0$.*

Then we have the continuous imbedding

$$N_p^\alpha(\Omega)\subseteq N_q^\beta(\Omega). \tag{2.6}$$

Proof. For a proof the reader is directed to [16, pp. 236–237, 381].

The nice thing about (2.6) is that, given a function in a certain L_p-Nikol'skii space, we can, from any $q>p$ calculate very easily which L_q-Nikol'skii space that function naturally lies in. When $q>p$, we have $\beta<\alpha$ and so, in effect, we are obtaining extra integrability at the cost of giving away some smoothness. The parallel of this process, using (2.4), is that estimates for the L_p-modulus of smoothness of any function may be used to infer estimates for the L_q-modulus, for all $q>p$. This observation is very useful in the developments of Section 3, where we estimate the L_1-modulus of a class of functions and then apply the above observations to obtain results about the L_q-modulus for any $q > 1$, without any further painful calculations.

We are now ready for the main results of this paper.

3. The Main Results

The main results depend on a technical lemma (Lemma 1) which is stated and proved below.

From now on, let $A = \{a_1, ..., a_m\}$ be a fixed subset of $\bar{\Omega}$. For $t \in \Omega$, we define the function

$$\sigma_A(t) = \min_{j=1,...,m} |t - a_j|.$$

Thus $\sigma_A(t)$ is the *distance* of t from A.

For any $\delta > 0$ and $t \in \mathbb{R}^n$, the open ball centred on t with radius δ will be denoted by $B(t, \delta)$. The set $\bigcup_{j=1}^m B(a_j, \delta)$ will be denoted by $B(A, \delta)$. Throughout the remainder of this paper, C will denote a generic positive constant which may depend on various quantities at various times, but will never depend on h or τ.

Lemma 1. *Let $h \in \mathbb{R}^n$, and let $f \in L_1(\Omega)$.*

(i) *Suppose that for $|i| = 1$, $D^i f \in C(\Omega \backslash A)$. Then*

$$\|\Delta_h f\|_{1,\Omega_h} \leqslant C\left[\|f\|_{1,\Omega \cap B(A,2|h|)} + |h| \sum_{|i|=1} \int_0^1 \|D^i f\|_{1,\Omega \backslash B(A,\lambda|h|)}\, d\lambda\right],$$

provided the second term on the right-hand side is a convergent repeated integral.

(ii) *Suppose the conditions of* (i) *are satisfied for $|i| = 2$ also. Then*

$$\|\Delta_h^2 f\|_{1,\Omega_{2h}} \leqslant C\left[\|f\|_{1,\Omega \cap B(A,4|h|)} + |h|^2 \sum_{|i|=2} \int_0^1 \int_0^1 \|D^i f\|_{1,\Omega \backslash B(A,(\lambda+\mu)|h|)}\, d\mu\, d\lambda\right],$$

provided the second term on the right-hand side is a convergent repeated integral.

Proof. (i) Observe first that

$$\|\Delta_h f\|_{1,\Omega_h} = \|\Delta_h f\|_{1,\Omega_h \cap B(A,|h|)} + \|\Delta_h f\|_{1,\Omega_h \backslash B(A,|h|)}. \tag{3.1}$$

Now, on use of the inequality

$$|\Delta_h f(t)| \leqslant |f(t+h)| + |f(t)|,$$

it follows easily that

$$\|\Delta_h f\|_{1,\Omega_h \cap B(A,|h|)} \leqslant 2\|f\|_{1,\Omega \cap B(A,2|h|)}. \tag{3.2}$$

Also,

$$\|\Delta_h f\|_{1,\Omega_h \backslash B(A,|h|)} = \int_{\Omega_h \backslash B(A,|h|)} \left| \int_0^1 \frac{\partial}{\partial \lambda} f(t+\lambda h)\, d\lambda \right| dt$$

$$\leqslant |h| \sum_{|i|=1} \int_{\Omega_h \backslash B(A,|h|)} \int_0^1 |D^i f(t+\lambda h)|\, d\lambda\, dt. \tag{3.3}$$

We then use Fubini's theorem to reverse the order of integration in (3.3). The observation that $\{t+\lambda h: t \in \Omega_h \backslash B(A, |h|)\} \subseteq \Omega \backslash B(A, (1-\lambda)|h|)$ for all $\lambda \in (0, 1)$ then yields

$$\|\Delta_h f\|_{1,\Omega_h \backslash B(A,|h|)} \leqslant |h| \sum_{|i|=1} \int_0^1 \int_{\Omega \backslash B(A,(1-\lambda)|h|)} |D^i f(t)|\, dt\, d\lambda$$

$$= |h| \sum_{|i|=1} \int_0^1 \int_{\Omega \backslash B(A,\lambda|h|)} |D^i f(t)|\, dt\, d\lambda, \tag{3.4}$$

where the final inequality arises simply from the change of variable $\lambda' = 1 - \lambda$. Substitution of (3.2) and (3.4) in (3.1) yields the required result.

(ii) The proof is analogous to (i). Note first that

$$\|\Delta_h^2 f\|_{1,\Omega_{2h}} = \|\Delta_h^2 f\|_{1,\Omega_{2h} \cap B(A,2|h|)} + \|\Delta_h^2 f\|_{1,\Omega_{2h} \backslash B(A,2|h|)}. \tag{3.5}$$

Then (cf. (3.2))

$$\|\Delta_h^2 f\|_{1,\Omega_{2h} \cap B(A,2|h|)} \leqslant 4\, \|f\|_{\Omega \cap B(A,4|h|)}. \tag{3.6}$$

Since D commutes with Δ, two successive applications of the technique used to prove (3.3) will yield

$$\|\Delta_h^2 f\|_{1,\Omega_{2h} \backslash B(A,2|h|)}$$

$$\leqslant |h| \sum_{|i|=1} \int_{\Omega_{2h} \backslash B(A,2|h|)} \int_0^1 |\Delta_h D^i f(t+\lambda h)|\, d\lambda\, dt$$

$$\leqslant |h|^2 \sum_{|i|=2} \int_{\Omega_{2h} \backslash B(A,2|h|)} \int_0^1 \int_0^1 |D^i f(t+\lambda h+\mu h)|\, d\mu\, d\lambda\, dt. \tag{3.7}$$

Then, using Fubini's theorem and the observation that

$$\{t+\lambda h+\mu h: t\in\Omega_{2h}\backslash B(A, 2\,|h|)\}\subseteq\Omega\backslash B(A, (2-\lambda-\mu)\,|h|),$$

for all $(\lambda, \mu)\in(0, 1)\times(0, 1)$, we obtain from (3.7),

$$\|\Delta_h^2 f\|_{1,\Omega_{2h}\backslash B(A,2|h|)}$$
$$\leqslant |h|^2 \sum_{|i|=2}\int_0^1\int_0^1\int_{\Omega\backslash B(A,(2-\lambda-\mu)|h|)} |D^i f(t)|\,dt\,d\mu\,d\lambda.$$

Using the changes of variable $\lambda'=(1-\lambda)$, $\mu'=(1-\mu)$ and substituting, along with (3.6) in (3.5) yields the result.

As we shall see in Lemma 2, the results of Lemma 1 allow us to estimate readily the modulus of smoothness of certain classes of functions. The approach used here is adapted from a method of Kantorovich and Akilov [14, pp. 362–365] (which was later used by Pitkäranta [17] and the author [9]) to attack a rather different problem, namely the analysis of multidimensional singular integrals.

We shall be concerned mainly with the following class of functions.

Definition. For $\mu\in\mathbb{R}$, $l\in\mathbb{N}$, we say that $f: \Omega\to\mathbb{C}$ is in the class $K(\mu, l)$ if for $|i|=0,..., l$, $D^i f\in C(\Omega\backslash A)$ and

$$|D^i f(t)|\leqslant C(\sigma_A(t))^{\mu-|i|}, \qquad t\in\Omega\backslash A,$$

with C independent of i and t. When $\mu>0$ we also assume that $f\in C^{[\mu]}(\bar{\Omega})$.

Classes of functions similar to the above class have been considered by other authors (e.g., de Boor and Rice [3]).

We remark that if $f\in K(\mu, l)$ and $|\beta|=0,..., l$, we then have $D^\beta f\in K(\mu-|\beta|, l-|\beta|)$. Also, if $f\in K(\mu, l)$ and if $0\leqslant r\leqslant l$ can be chosen with $r<\mu+n$, then it follows that $f\in H_1^r(\Omega)$.

Examples of typical functions in $K(\mu, l)$ are easily identified. Consider the simple case when $A=\{\theta\}\subset\bar{\Omega}$. Then the function $|t|^\mu$ is in $K(\mu, l)$ for all $l\in\mathbb{N}$, $\mu\in\mathbb{R}$. Perhaps a more illuminating example when $\mu=0$ is the function $t_i/|t|$ (where t_i is the ith component of the vector t), which is in $K(0, l)$ for all $l\in\mathbb{N}$. For $n=1$ the function $|t|^\mu/(\ln|t|)$ is in $K(\mu, l)$ for all $l\in\mathbb{N}$, $\mu\in\mathbb{R}$.

Lemma 2. (i) *Let* $0<\alpha\leqslant 1$, *and let* $f\in K(\alpha-n, 1)$. *Then*

$$\omega_1(f, \tau)_1=\begin{cases}O(\tau^\alpha) & (0<\alpha<1),\\ O(\tau\,|\ln\tau|) & (\alpha=1).\end{cases}$$

(ii) *Let* $f \in K(1-n, 2)$. *Then*

$$\omega_2(f, \tau)_1 = O(\tau).$$

Proof. Let $h \in \mathbb{R}^n$ with $|h| \leqslant D =$ diameter of Ω. (Note that if $|h| > D$, then $\Omega_h = \varnothing = \Omega_{2h}$, and $\|\Delta_h f\|_{\Omega_h}$ and $\|\Delta_h^2 f\|_{\Omega_{2h}}$ will be just zero.) We have

$$\|f\|_{1,\Omega \cap B(A,2|h|)} = \sum_{j=1}^{m} \int_{\Omega_j} |f(t)| \, dt,$$

where, for $j = 1,..., m$, $\Omega_j = \{t \in \Omega \cap B(A, 2\,|h|)\colon\ \sigma_A(t) = |t - a_j|\}$. Then $\Omega_j \subseteq B(a_j, 2\,|h|)$. (To see this, let $t \in \Omega_j$. Then $t \in B(a_i, 2\,|h|)$ for some $i = 1,..., m$, and $\sigma_A(t) = |t - a_j|$. Hence $|t - a_j| \leqslant |t - a_i| < 2\,|h|$, yielding $t \in B(a_j, 2\,|h|)$.) Thus it follows that

$$\|f\|_{1,\Omega \cap B(A,2|h|)} \leqslant C \sum_{j=1}^{m} \int_{B(a_j,2|h|)} |t - a_j|^{\alpha - n} \, dt. \tag{3.8}$$

Also, if $|i| = 1$,

$$\|D^i f\|_{1,\Omega \backslash B(A,\lambda|h|)} \leqslant C \sum_{j=1}^{m} \int_{A_j} |t - a_j|^{\alpha - n - 1} \, dt,$$

where, for $j = 1,..., m$, $A_j = \{t \in \Omega \backslash B(A, \lambda\,|h|)\colon\ \sigma_A(t) = |t - a_j|\}$. Clearly $A_j \subseteq \Omega \backslash B(a_j, \lambda\,|h|)$, and so

$$\|D^i f\|_{1,\Omega \backslash B(A,\lambda|h|)} \leqslant C \sum_{j=1}^{m} \int_{\Omega \backslash B(a_j,\lambda|h|)} |t - a_j|^{\alpha - n - 1} \, dt. \tag{3.9}$$

Using spherical polar coordinates with origin a_j in (3.8) and (3.9), we obtain

$$\|f\|_{1,\Omega \cap B(A,2|h|)} \leqslant C \int_0^{2|h|} r^{\alpha - 1} \, dr \leqslant C\,|h|^\alpha, \tag{3.10}$$

and

$$\|D^i f\|_{1,\Omega \backslash B(A,\lambda|h|)} \leqslant C \int_{\lambda|h|}^{D} r^{\alpha - 2} \, dr. \tag{3.11}$$

Integrating with respect to r in (3.11), and then again with respect to λ, we obtain

$$|h| \sum_{|i|=1} \int_0^1 \|D^i f\|_{1,\Omega \backslash B(A,\lambda|h|)} \, d\lambda \leqslant \begin{cases} C\,|h|^\alpha & (\alpha \neq 1), \\ C\,|h|\,|\ln |h|| & (\alpha = 1). \end{cases} \tag{3.12}$$

Combination of (3.10) and (3.12) in the estimate of Lemma 1(i) yields the required result.

The proof of (ii) uses Lemma 1(ii) and follows similar lines to the proof of (i).

Remark 1. Let $n=1$, $\Omega=(-1,1)$ and consider the function $f(t)=\ln|t|$. (Thus $A=\{\theta\}$, in this case.) Then it can be shown (see below for details) that

$$\omega_1(f,\tau)_1 = O(\tau\,|\ln\tau|) \tag{3.13}$$

and

$$\omega_2(f,\tau)_1 = O(\tau). \tag{3.14}$$

Looking at this example and the results of Lemma 2 in the case $\alpha=1$ we see that the class of functions for which (3.14) is satisfied is, in a sense, more natural than the class for which

$$\omega_1(f,\tau)_1 = O(\tau).$$

Further evidence for this assertion can be found in [20], where the class of functions satisfying (3.14) is studied.

The proof of (3.13) is straightforward: Arguments analogous to those which obtained (3.10) and (3.12) give

$$\left.\begin{array}{r} \|f\|_{1,\Omega\cap B(\theta,2|h|)} \\ |h|\displaystyle\int_0^1 \|f'\|_{\Omega\setminus B(\theta,\lambda|h|)}\,d\lambda \end{array}\right\} = O(|h|\,|\ln|h||),$$

and (3.13) follows from Lemma 1(i).

To prove (3.14), we note first that calculation analogous to the above will yield

$$\|f\|_{1,\Omega\cap B(\theta,4|h|)} = O(|h|\,|\ln|h||)$$

and

$$|h|^2\int_0^1\int_0^1 \|f''\|_{1,\Omega\setminus B(\theta,(\lambda+\mu)|h|)}\,d\mu\,d\lambda = O(|h|).$$

Hence we cannot use Lemma 1(ii) directly to obtain (3.14). Instead, we use the sharper estimate

$$\|\Delta_h^2 f\|_{1,\Omega_{2h}} \leqslant \|\Delta_h^2 f\|_{1,\Omega_{2h}\cap B(\theta,2|h|)} + |h|^2\int_0^1\int_0^1 \|f''\|_{1,\Omega\setminus B(\theta,(\lambda+\mu)|h|)}\,d\mu\,d\lambda,$$

which follows from (3.5). As observed above, the second term on the right-hand side is $O(|h|)$. Also, it can be shown by elementary calculations that the first term is $O(|h|)$ also, and so (3.14) follows.

Remark 2. The arguments of Lemma 2 may be extended to treat more general types of singularities. For example, consider the case when $k \in \mathbb{N}_0$, $l \in \mathbb{N}$, $0 < \alpha \leqslant 1$, and f satisfies, for $|i| = 0, 1, ..., l$,

$$D^i f \in C(\Omega \backslash A) \quad \text{with} \quad |D^i f(t)| \leqslant C(\ln(\sigma_A(t)))^k (\sigma_A(t))^{\alpha - n - |i|}, \tag{3.15}$$

Then if (3.15) is satisfied for $l = 1$, arguments analogous to Lemma 2(i) show that

$$\omega_1(f, \tau)_1 = \begin{cases} O(\tau^\alpha \, |\ln \tau|^k) & (0 < \alpha < 1), \\ O(\tau \, |\ln \tau|^{k+1}) & (\alpha = 1), \end{cases}$$

whereas arguments analogous to Lemma 2(ii) show that if $\alpha = 1$ and (3.15) is satisfied for $l = 2$, then

$$\omega_2(f, \tau)_1 = O(\tau \, |\ln \tau|^k).$$

Recall now the notation (2.5).

THEOREM 3. (i) *Let* $\alpha > 0$, *and let* $f \in K(\alpha - n, [\alpha] + 1)$. *Then if* $k \in \mathbb{N}$,

$$\omega_k(f, \tau)_1 = \begin{cases} O(\tau^k), & k < \alpha, \\ O(\tau^\alpha), & k > \alpha \notin \mathbb{N}, \\ O(\tau^\alpha \, |\ln \tau|), & k \geqslant \alpha \in \mathbb{N}. \end{cases}$$

(ii) *Let* $\alpha \in \mathbb{N}$, *and let* $f \in K(\alpha - n, [\alpha] + 2)$. *Then if* $k \in \mathbb{N}$,

$$\omega_k(f, \tau)_1 = \begin{cases} O(\tau^k), & k < \alpha, \\ O(\tau^\alpha \, |\ln \tau|), & k = \alpha, \\ O(\tau^\alpha), & k > \alpha. \end{cases}$$

Proof. First, note (see the discussion following the definition of $K(\mu, l)$) that, under the conditions of either (i) or (ii), we have $f \in H_1^{[\alpha]}(\Omega)$. Hence for $k < \alpha$, we have $k \leqslant [\alpha]$ and so use of (2.2) yields

$$\omega_k(f, \tau)_1 = O(\tau^k),$$

proving the first estimate for each of (i) and (ii). We now prove the remaining estimates.

(i) When $k \in \mathbb{N}$ and $k > \alpha \notin \mathbb{N}$ or $k \geqslant \alpha \in \mathbb{N}$, it follows easily that $k \geqslant [\alpha]+1$. Hence (2.1), (2.2), and Lemma 2(i) yield

$$\begin{aligned}
\omega_k(f, \tau)_1 &\leqslant C\tau^{[\alpha]} \sup_{|\beta|=[\alpha]} \omega_1(D^\beta f, \tau)_1 \\
&\leqslant \begin{cases} C\tau^{[a]}\tau^{\alpha_0} & (k > \alpha \notin N), \\ C\tau^{[\alpha]}\tau^{\alpha_0} |\ln \tau| & (k \geqslant \alpha \in N), \end{cases} \\
&= \begin{cases} O(\tau^\alpha) & (k > \alpha \notin N), \\ O(\tau^\alpha |\ln \tau|) & (k \geqslant \alpha \in N). \end{cases}
\end{aligned}$$

The second inequality follows from Lemma 2(i), since $D^\beta f \in K(\alpha_0 - n, 1)$ for $|\beta| = [\alpha]$, and $\alpha_0 < 1$ when $\alpha \notin \mathbb{N}$, whilst $\alpha_0 = 1$ when $\alpha \in \mathbb{N}$.

(ii) Since $K(\alpha - n, [\alpha]+2) \leqslant K(\alpha - n, [\alpha]+1)$, the second estimate of (ii) has already been proved in (i). For the third estimate, let $k > \alpha \in \mathbb{N}$, we have $k \geqslant [\alpha]+2$ and so (2.1), (2.2), and Lemma 2(ii) yield

$$\begin{aligned}
\omega_k(f, \tau)_1 &\leqslant C\tau^{[\alpha]} \sup_{|\beta|=[\alpha]} \omega_2(D^\beta f, \tau)_1 \\
&\leqslant C\tau^{[\alpha]}\tau \\
&= O(\tau^\alpha).
\end{aligned}$$

Again, the seond inequality follows from Lemma 2(ii) since $D^\beta f \in K(1-n, 2)$, for $|\beta| = [\alpha]$.

In the following corollary, the results of Theorem 3 are used to obtain typical elements of the L_1-Nikol'skii spaces introduced in Section 2. Assume from now on that condition (i) of Nikol'skii's imbedding theorem is satisfied.

COROLLARY 4. *Let $\alpha > 0$. If either*

$$f \in K(\alpha - n, [\alpha]+1) \qquad (\alpha \notin \mathbb{N})$$

or

$$f \in K(\alpha - n, [\alpha]+2) \qquad (\alpha \in \mathbb{N}),$$

then $f \in N_1^\alpha(\Omega)$.

Proof. Let $k \in \mathbb{N}$ with $k > \alpha$. Then Theorem 3 implies that

$$\omega_k(f, \tau)_1 = O(\tau^\alpha).$$

Since $(k, 0)$ is an admissible pair for α, it follows that $f \in N_1^\alpha(\Omega)$.

The final result of this section uses Nikol'skii's imbedding theorem and the results that we have already obtained above to calculate the modulus of smoothness of certain classes of functions with respect to a general L_q-norm.

THEOREM 5. *Let* $\alpha > 0$, $1 \leqslant q \leqslant \infty$ *and suppose* $f \in K(\alpha - n/q, [\alpha - n/q + n] + 2)$. *Suppose also that hypothesis* (i) *of Nikol'skii's imbedding theorem is satisfied. Then for* $k \in \mathbb{N}$ *we have*

$$\omega_k(f, \tau)_q = \begin{cases} O(\tau^\alpha), & k > \alpha, \\ O(\tau^k), & k < \alpha. \end{cases}$$

Remark. When $k = \alpha$ various possibilities exist, including the generation of logarithmic terms. We omit these for simplicity.

Proof. Since

$$f \in K((\alpha - n/q + n) - n, [\alpha - n/q + n] + 2),$$

and since

$$\alpha - n/q + n \geqslant \alpha > 0,$$

it follows by Corollary 4 that $f \in N_1^\eta(\Omega)$, with $\eta = \alpha - n/q + n$. Now, since

$$0 < \alpha = (\alpha - n/q + n) - n(1 - 1/q),$$

it follows by Nikol'skii's imbedding theorem that $f \in N_q^\alpha(\Omega)$. The first estimate follows since if $k > \alpha$, then $(k, 0)$ is an admissible pair for α.

Now let $k \in \mathbb{N}$, $k < \alpha$. Then

$$D^\beta f \in K(\alpha - k - n/q, [\alpha - k - n/q + n] + 2),$$

for all $|\beta| = k$. Since when $1 \leqslant q < \infty$, we have

$$(\alpha - k - n/q)q = (\alpha - k)q - n > -n,$$

it follows that $D^\beta f \in L_q(\Omega)$ and hence that $f \in W_q^k(\Omega)$. When $q = \infty$, we have

$$\alpha - k - n/q = \alpha - k > 0,$$

and so $D^\beta f \in C(\bar{\Omega})$. Hence $f \in C^k(\bar{\Omega})$. Thus overall $f \in H_q^k(\Omega)$, and the second estimate follows from (2.2).

An example of a function satisfying the conditions of Theorem 5, and for which the order of the modulus of smoothness is well known, is the Bessel potential kernel [19]. In turns out that the order of the modulus of smoothness of this function is exactly as predicted in Theorem 5. Further evidence of the sharpness of the estimates given here is provided in the next section.

4. Numerical Illustrations

Example 1. Let $f(t) = t^{3/4}$, for $t \in (0, 1)$. Then it is easy to show that $f \in K(\frac{3}{4}, l)$ for all $l \in \mathbb{N}$, where in this case $A = \{0\}$. Writing $\frac{3}{4} = \alpha - 1/q$, and calculating α, Theorem 5 then predicts that

$$\omega_k(f, \tau)_q = O(\tau^\rho), \tag{4.1}$$

where ρ is given for various values of k and q in Table I. The entry 1* in Table I indicates that Theorem 5 actually predicts that

$$\omega_1(f, \tau)_4 = O(\tau^\rho) \qquad \text{for all} \quad \rho < 1.$$

The quantities $\omega_k(f, \tau)_q$ were approximated six times using $\tau_i = 0 \cdot 2/2^i$, for $i = 4, 5, 6, 7, 8, 9$, and the number ρ estimated five times (one estimate for each pair of consecutive values of τ) using the formula

$$\rho = \frac{\ln(\omega_k(f, \tau_i)_q / \omega_k(f, \tau_{i+1})_q)}{\ln(2)}, \tag{4.2}$$

for $i = 4, ..., 8$. The five values of ρ thus obtained for each k and q are shown in order of increasing i in Table II. The figures are rounded to the given number of significant figures.

Example 2. Let $f(t) = |t|^{1/2}$, $t \in (0, 1) \times (0, 1)$. Then $f \in K(\frac{1}{2}, l)$, for all $l \in \mathbb{N}$, where $A = \{(0, 0)\}$. Writing $\frac{1}{2} = \alpha - 2/q$ and calculating α, Theorem 5 then predicts that (4.1) holds where ρ is given for various values of k and q in Table III. In this case the quantities $\omega_k(f, \tau)_q$ were approximated five

TABLE I

	q			
k	1	2	3	4
1	1	1	1	1*
2	$\frac{7}{4}$	$\frac{5}{4}$	$\frac{13}{12}$	1

TABLE II

k	q = 1	2	3	4
1	0·983784	0·981703	0·971601	0·954824
	0·990768	0·987579	0·977512	0·959944
	0·994699	0·991465	0·981948	0·963990
	0·996938	0·994084	0·985365	0·967279
	0·998222	0·995876	0·988047	0·970010
2	1·64366	1·24952	1·08333	1·00000
	1·66674	1·24983	1·08333	1·00000
	1·68371	1·24994	1·08333	1·00000
	1·69660	1·24998	1·08333	1·00000
	1·70660	1·24999	1·08333	1·00000

times using $\tau_i = 0 \cdot 2/2^i$, for $i = 4, 5, 6, 7, 8$, and the number ρ estimated four times (one estimate for each pair of consecutive values of τ) using (4.2), for $i = 4, \ldots, 7$. The four values of ρ thus obtained for each k and q are given in Table IV. The last two values of ρ for $k = 2$, $q = 3$ were contaminated by rounding error. Note that in this case, since we are working in two dimensions, we have

$$\omega_k(f, \tau)_q = \sup_{0 < |h| \leqslant \tau} \|\Delta_h^k f\|_{q,\Omega_{kh}},$$

where the supremum is taken over all $h \in \mathbb{R}^2$ with $0 < |h| \leqslant \tau$. In this case the singularity at f occurs at the bottom left-hand corner of $\bar{\Omega} = [0, 1] \times [0, 1]$ (i.e., at the point $(0, 0)$). Thus, since we would expect the behaviour of $\omega_k(f, \tau)_q$ to be dominated by the quantity $\|\Delta_h^k f\|_{\Omega_{kh}}$ when $\bar{\Omega}_{kh}$ actually *contains* the singularity of f, we have restricted attention to the case when $h \in \{(x, y) \in \mathbb{R}^2 \colon x \geqslant 0,\ y \geqslant 0\}$. As well, since the domain is symmetrical about $x = y$, we have confined attention to the case when $h \in \{(x, y) \in \mathbb{R}^2 \colon x \geqslant 0,\ y \geqslant 0,\ x \geqslant y\}$. Accordingly, we have estimated

TABLE III

k	q = 1	2	3
1	1	1	1
2	2	$\frac{3}{2}$	$\frac{7}{6}$

TABLE IV

	q		
k	1	3	3
1	0·987358	0·990803	0·982312
	0·993682	0·995062	0·987668
	0·996840	0·997353	0·991600
	0·998419	0·998589	0·994727
2	1·89152	1·48901	1·17189
	1·92568	1·49454	1·16200
	1·94857	1·49792	1·84339
	1·96419	1·49832	—

$\omega_k(f, \tau)_q$ by calculating $\|\Delta_h^k f\|_{q,\Omega_{kh}}$ for each h in the "fan" $\{(x, y): x = \tau\cos(j\pi/20),\ y = \tau\sin(j\pi/20),\ j = 0, \ldots, 5\}$, and then taking the supremum over these six values of h.

All numerical calculations were done in double precision on the VAX VMS 11/780 at the University of Melbourne. The numerical integrations necessary for calculation of the moduli of smoothness were done using adaptive integration packages from the IMSL library. We used DCADRE for the one-dimensional case (Example 1) and DBLINT for the two-dimensional case (Example 2).

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